1) a) $X_n \uparrow$ $\quad X_n \xrightarrow{P} X \iff X_n \xrightarrow{a.s.} X$

$\Leftarrow$ always.

Proof Idea: $X_n \xrightarrow{P} X \Rightarrow X_{n_k} \to X$ a.s.

If $n > n_k \Rightarrow X - X_n < X - X_{n_k} \to 0$

b) $X_n \xrightarrow{a.s.} X \iff \sup_{k \geq n} |X_k - X| \xrightarrow{P} 0$

$|X_n - X| \xrightarrow{a.s.} 0$ $\quad$ A.e. x $\forall \varepsilon > 0 \; \exists N;\; n > N \; |X_n(x) - X(x)| < \varepsilon$

sup

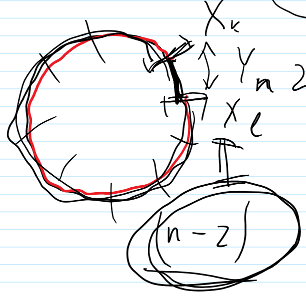
$\sup_{k \geq n} |X_k - X| \xrightarrow{a.s.} 0$

$Y_n = \sup_{k \geq n} |X_k - X| \downarrow \iff Y_n \xrightarrow{P} 0$

$Y_n \xrightarrow{a.s.} 0$

c) $Y_n \downarrow$ $\quad Y_n \xrightarrow{a.s.} 0 \iff Y_n \xrightarrow{P} 0.$

? $\forall \varepsilon > 0 \quad P(Y_n > \varepsilon) \to 0$



$2\pi - \varepsilon = |I|$

$X_1, X_2, \dots, X_n \in I$

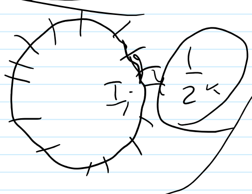
$P(X_1 \in I) = \frac{2\pi - \varepsilon}{2\pi} = 1 - \frac{\varepsilon}{2\pi}$

$P(X_1 \dots X_n \in I) = \left(1 - \frac{\varepsilon}{2\pi}\right)^{n-2}$

X_k, X_l

$P(Y_n > \varepsilon) \leq \left(1 - \frac{\varepsilon}{2\pi}\right)^{n-2}$ What did I forget? $\binom{n}{2}$

$Y_n > \varepsilon$ $\quad$ Fix k: $\frac{1}{2^k} < \frac{\varepsilon}{4}$



If $Y_n > \varepsilon$ then $\exists I_j : X_1 \notin I_j,\; X_2 \notin I_j,\; \dots,\; X_n \notin I_j$

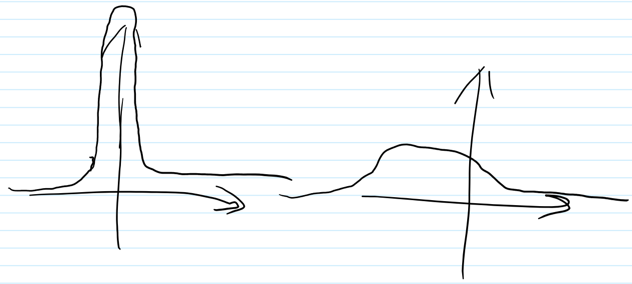
$$P(X_1 \notin I_j, \dots X_n \notin I_j) = \left(1 - \frac{1}{2^k}\right)^n \qquad X_n \notin I_j$$

$$P(\exists j : \dots) \le 2^k \left(1 - \frac{1}{2^k}\right)^n \to 0.$$

$$\vee\!\!= \quad P(Y_n \ge \varepsilon)$$

4) $E(X_i X_j) = \begin{cases} 0, & i \ne j \\ \sigma^2, & i = j \end{cases}$

7) $\sup_n \sigma_n < \infty$



9) $\exists \lambda: X = \lambda Y$ a.s

$0 \le E((X - \lambda Y)^2)$

11) $M_n = \max_{i \le n} X_i$ i.i.d

a) $P(M_n > x) \le n P(X_1 > x)$

$\{M_n > x\} = \bigcup_{j=1}^{n} \{X_j > x\}$

$P(\{M_n > x\}) \le \Sigma P\{X_j > x\} = n P(X_1 > x)$

c) $M_n/n \xrightarrow{P} 0 \iff n P(X_1 > n) \to 0$

$(\Rightarrow)$ $\forall \varepsilon > 0$ $P(M_n > n\varepsilon) \to 0$

$$P(\bigcup (X_k > n\varepsilon)) = 1 - P(\bigcap (X_k < n\varepsilon)) = 1 - (P(X_1 < n\varepsilon))^n$$

$(P(X_1 < n\varepsilon))^n \longrightarrow 1$

$$1-(P(X_1<n\varepsilon))$$

$$(P(X_1<n\varepsilon))^n \to 1$$

$$(1-P(X_1>n\varepsilon))^n \to 1$$

$$\left(1-P(X_1>n\varepsilon)\right)^n$$

$$n\log(1-P(X_1>n\varepsilon)) \to 0$$

$$\frac{\log(1-a)}{a} \xrightarrow[a\to 0]{} 1 \qquad nP(X_1>n\varepsilon) \to 0$$

$$\varepsilon = 1.$$

$$\Leftarrow \quad nP(X_1>n) \xrightarrow[n\to\infty]{} 0 \iff yP(X_1>y) \xrightarrow[y\to\infty]{} 0$$

$$n \le y < n+1$$

$$yP(X_1>y) \le (n+1)P(X_1>n) = \underbrace{\frac{n+1}{n}}_{\to 1}\underbrace{nP(X_1>n)}_{\to 0}$$

$$nP(X_1>n) \to 0 \Rightarrow n\varepsilon P(X_1>n\varepsilon) \to 0.$$

$$P\left(\frac{M_n}{n}>\varepsilon\right) = P(M_n>n\varepsilon) \underset{(a)}{\le} \underbrace{nP(X_1>n\varepsilon)}_{\downarrow\, 0}$$

$$yP(X_1>y) \to 0 \iff y_k P(X_1>y_k) \to 0$$

for $y_k \uparrow \infty$ $\sup\left(\frac{y_{k+1}}{y_k}\right) < \infty$

14). BC

$$\sum P(X_k = k^2) < \infty \Rightarrow \text{A.S. } \exists N: n>N$$

$$X_n = -1.$$

$$\sum X_k \to -\infty$$

17), 19) $\{Y_n>\varepsilon\} \supset \{X_n>\varepsilon\}$

25). a) Look at the proof that every

25). c) Look at the proof that every sequence ~~convergent~~ in P contains subsequence convergent a.s.

d) Assume: A.S convergence metrizable

i.e. $\exists\, d(X,Y)$: $X_n \xrightarrow{a.s.} X \iff d(X_n, X) \to 0$

Take any $X_n \to X$ in Probability but not a.s

$d(X_n, X) \not\to 0$.

$\exists\, X_{n_k}\ \exists \delta > 0\ d(X_{n_k}, X) > \delta\ \ \forall k$.

so $\exists\, X_{n_{k_\ell}} \xrightarrow{a.s.} X$, but $d(X_{n_{k_\ell}}, X) > \delta$

contradiction.

30) $E(|XY|) \le \|X\|_1 \|Y\|_\infty =$

$E(|X|)\ \text{ess sup}\, |Y|$.

$\text{ess sup}\, X = \sup\{x : P(X > x) > 0\}$.

essential supremum

$= \inf\{x : P(X > x) = 0\}$

$\bigcap_{p<\infty} L^p \neq L^\infty$, unless P is discrete

BMO

Bounded Mean Oscillation

P-Lebesgue on [0,1]

$E(|X1_I - E(X1_I)|) < \infty$

$\forall I \subset [0,1]$

interval